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Some Determinants Involving Incomplete Fubini Numbers

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Abstract

We study some properties of restricted and associated Fubini numbers. In particular, they have the natural extensions of the original Fubini numbers in the sense of determinants. We also introduce modified Bernoulli and Cauchy numbers and study characteristic properties.

1 Introduction

The Fubini numbers (or the ordered Bell numbers) form an integer sequence in which the nth term counts the number of weak orderings of a set with n elements. The Stirling numbers of the second kind $S_2(n,k)$ counts the number of ways that the set $A=\{1,2,\dots,n\}$ with n elements can be partitioned into k non-empty subsets. Fubini numbers are defined by

$$F_n=\sum_{k=0}^{n}k!S_2(n,k)$$

([18]). They can be expanded involving binomial coefficients or given by an infinite series.

$$F_n=\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n=\frac{1}{2}\sum_{m=0}^{\infty}\frac{m^n}{2^m}\,.$$

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The first Fubini numbers are presented as

$$\{F_n\}_{n=0}^{\infty} = \{1, 1, 3, 13, 75, 541, 4683, 47293, 545835, 7087261, 102247563, \dots\}$$

([22, A000670]). The (exponential) generating function of Fubini numbers is given by

$$\frac{1}{2 - e^t} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!}\,. \tag{1}$$

The Fubini numbers satisfy the recurrence relation:

$$F_n = \sum_{j=1}^{n} \binom{n}{j} F_{n-j}\,. \tag{2}$$

The literature contains several generalizations of Stirling numbers; see [16]. Among them, the so-called restricted and associated Stirling numbers of both kinds (cf. [3, 6, 7, 10, 11, 12, 18, 20]). The restricted Stirling number of the second kind $S_2\,(n,k)_{\leq m}$ gives the number of partitions of n elements into k subsets under the restriction that none of the blocks contain more than m elements. The associated Stirling number of the second kind $S_2\,(n,k)_{\geq m}$ gives the number of partitions of an n element set into k subsets such that every block contains at least m elements.

In this paper, we are interested in the sequence of the number of weak orderings of a set with n elements in blocks with at most (and at least) m elements. These two integer sequences are called the restricted Fubini numbers and associated Fubini numbers, respectively. In particular, we give some new determinant expressions for these sequences, and for some related sequences such as the modified Bernoulli and Cauchy numbers. We also apply the Trudi's formula to obtain new explicit combinatorial formulas for these sequences.

Recently, studies on the determinantal representations involving interesting combinatorial sequences appeared. For example, Bozkurt and Tam [4] established formulas for the determinants and inverses of r-circulant matrices of second-order linear recurrences. Li and MacHenry [14] gave a general method for finding permanental and determinantal representations of many families of integer sequences such as Fibonacci and Lucas polynomials. In [15], Li obtained three new Fibonacci-Hessenberg matrices and studied its relations with Pell and Perrin sequence. Xu and Zhou [26] studied a determinantal representation for a generalization of the Stirling numbers, called r-Whitney numbers, see also [19]. More examples can be found in [23, 24, 27].

2 Determinants

In [8], Glaisher gave determinant expressions of several numbers, including Bernoulli numbers B_n, Cauchy numbers c_n and Euler numbers E_n, defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\quad \frac{t}{\ln(1+t)}=\sum_{n=0}^{\infty}c_n\frac{t^n}{n!}\quad\text{and}\quad \frac{1}{\cosh t}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},$$

respectively. A determinant expression of the classical Bernoulli numbers is given by

$$B_n=(-1)^n n!\begin{vmatrix} \frac{1}{2!} & 1 & & & \\ \frac{1}{3!} & \frac{1}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix}. \tag{3}$$

A determinant expression of Cauchy numbers is given by

$$c_n=n!\begin{vmatrix} \frac{1}{2} & 1 & & & \\ \frac{1}{3} & \frac{1}{2} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{3} & \frac{1}{2} \end{vmatrix}. \tag{4}$$

The value of this determinant, that is, $b_n=c_n/n!$ are called Bernoulli numbers of the second kind.

A determinant expression of Euler numbers is given by

$$E_{2n}=(-1)^n(2n)!\begin{vmatrix} \frac{1}{2!} & 1 & & & \\ \frac{1}{4!} & \frac{1}{2!} & 1 & & \\ \vdots & & \ddots & \ddots & \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & & \frac{1}{2!} & 1 \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}. \tag{5}$$

Several hypergeometric numbers can be also expressed in similar ways ([1, 2, 13]). However, all numbers cannot be expressed in such simpler determinant expressions. In general, we have the following relation:

Theorem 1. *Let $\{\alpha_n\}_{n\geq 0}$ be a sequence with $\alpha_0 = 1$, and $R(j)$ be a function independent of n. Then*

$$\alpha_n = \begin{vmatrix} R(1) & 1 & & & \\ R(2) & R(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ R(n-1) & R(n-2) & \cdots & R(1) & 1 \\ R(n) & R(n-1) & \cdots & R(2) & R(1) \end{vmatrix}. \tag{6}$$

if and only if

$$\alpha_n = \sum_{j=1}^{n} (-1)^{j-1} R(j)\alpha_{n-j} \quad (n \geq 1) \tag{7}$$

with $\alpha_0 = 1$.

Proof. When $n = 1$, the result is clear because $\alpha_1 = R(1)$. Assume that (7) is valid for any positive integer n. Suppose that (6) is true up to $n-1$. Then by expanding the first row, we have

$$\begin{vmatrix} R(1) & 1 & & & \\ R(2) & R(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ R(n-1) & R(n-2) & \cdots & R(1) & 1 \\ R(n) & R(n-1) & \cdots & R(2) & R(1) \end{vmatrix}$$

$$= R(1)\alpha_{n-1} - \begin{vmatrix} R(2) & 1 & & & \\ R(3) & R(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ R(n-1) & R(n-3) & \cdots & R(1) & 1 \\ R(n) & R(n-2) & \cdots & R(2) & R(1) \end{vmatrix}$$

$$= R(1)\alpha_{n-1} - R(2)\alpha_{n-2} + \begin{vmatrix} R(3) & 1 & & & \\ R(4) & R(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ R(n-1) & R(n-4) & \cdots & R(1) & 1 \\ R(n) & R(n-3) & \cdots & R(2) & R(1) \end{vmatrix}$$

$$= R(1)\alpha_{n-1} - R(2)\alpha_{n-2} + \cdots + (-1)^n \begin{vmatrix} R(n-1) & 1 \\ R(n) & R(1) \end{vmatrix}$$

$$= \sum_{j=1}^{n} (-1)^{j-1} R(j)\alpha_{n-j} = \alpha_n .$$

On the contrary, if (6) is true, by expanding the determinant, we get the relation (7). □

Fubini numbers satisfy the required relation. Therefore, we have a determinant expression of Fubini numbers similarly.

Theorem 2. *For $n \geq 1$, we have*

$$F_n = n! \begin{vmatrix} \frac{1}{1!} & 1 & & & \\ -\frac{1}{2!} & \frac{1}{1!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{(-1)^{n-2}}{(n-1)!} & \frac{(-1)^{n-3}}{(n-2)!} & \cdots & \frac{1}{1!} & 1 \\ \frac{(-1)^{n-1}}{n!} & \frac{(-1)^{n-2}}{(n-1)!} & \cdots & -\frac{1}{2!} & \frac{1}{1!} \end{vmatrix} .$$

Proof. From the relation (2), we apply Theorem 1 with

$$\alpha_n = \frac{F_n}{n!} \quad \text{and} \quad R(j) = \frac{(-1)^{j-1}}{j!} .$$

□

Examples. Since it is well-known that

$$\sum_{m=0}^{n} \binom{n+1}{m} B_m = 0 ,$$

we get

$$B_n = -\sum_{j=1}^{n} \frac{-n!}{(j+1)!(n-j)!} B_{n-j} .$$

Applying Theorem 1 with

$$\alpha_n = (-1)^n \frac{B_n}{n!} \quad \text{and} \quad R(j) = \frac{1}{(j+1)!} ,$$

we have the determinant expression (3).

Since

$$\frac{c_n}{n!} = \sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}}{n-k+1} \frac{c_k}{k!} = \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \frac{c_{n-j}}{(n-j)!}$$

([17, Theorem 2.1]), applying Theorem 1 with

$$\alpha_n = \frac{c_n}{n!} \quad \text{and} \quad R(j) = \frac{1}{j+1} ,$$

we have the determinant expression (4).

Since it is well-known that

$$\sum_{m=0}^{n} \binom{2n}{2m} E_{2m} = 0\,,$$

we get

$$(-1)^n \frac{E_{2n}}{(2n)!} = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(2j)!} \frac{(-1)^{n-j} E_{2(n-j)}}{(2n-2j)!}\,.$$

Applying Theorem 1 with

$$\alpha_n = (-1)^n \frac{E_{2n}}{(2n)!} \quad \text{and} \quad R(j) = \frac{1}{(2j)!}\,,$$

we have the determinant expression (5).

2.1 Applications of the Trudi's Formula

We can use the Trudi's formula to obtain an explicit formula for the numbers α_n in Theorem 1. This relation is known as Trudi's formula [21, Vol.3, p.214],[25] and the case $a_0 = 1$ of this formula is known as Brioschi's formula [5],[21, Vol.3, pp.208–209].

Theorem 3 (Trudi's formula [25])**.** *Let m be a positive integer. Then*

$$\begin{vmatrix} a_1 & a_2 & \cdots & & a_m \\ a_0 & a_1 & \cdots & & \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & \cdots & a_0 & a_1 \end{vmatrix} = \sum_{t_1+2t_2+\cdots+mt_m=m} \binom{t_1+\cdots+t_m}{t_1,\ldots,t_m} (-a_0)^{m-t_1-\cdots-t_m} a_1^{t_1} a_2^{t_2} \cdots a_m^{t_m}, \quad (8)$$

where $\binom{t_1+\cdots+t_m}{t_1,\ldots,t_m}$ is the multinomial coefficient.

From Trudi's formula, it is possible to give the combinatorial formula

$$\alpha_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{n-t_1-\cdots-t_n} R(1)^{t_1} R(2)^{t_2} \cdots R(n)^{t_n}.$$

In addition, there exists the following inversion formula, which follows from Theorem 1.

Lemma 4. *If $\{\alpha_n\}_{n\geq 0}$ is a sequence defined by $\alpha_0 = 1$ and*

$$\alpha_n = \begin{vmatrix} R(1) & 1 & & \\ R(2) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ R(n) & \cdots & R(2) & R(1) \end{vmatrix}, \text{ then } R(n) = \begin{vmatrix} \alpha_1 & 1 & & \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix}.$$

Moreover, if

$$A = \begin{bmatrix} 1 & & & \\ \alpha_1 & 1 & & \\ \vdots & \ddots & \ddots & \\ \alpha_n & \cdots & \alpha_1 & 1 \end{bmatrix}, \text{ then } A^{-1} = \begin{bmatrix} 1 & & & \\ R(1) & 1 & & \\ \vdots & \ddots & \ddots & \\ R(n) & \cdots & R(1) & 1 \end{bmatrix}.$$

From above relations and Theorem 2 we obtain an explicit formula for the Fubini sequence.

Corollary 5. *For $n \geq 1$*

$$\frac{F_n}{n!} = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{n-t_1-\cdots-t_n} \times \left(\frac{1}{1!}\right)^{t_1} \left(\frac{(-1)^1}{2!}\right)^{t_2} \cdots \left(\frac{(-1)^{n-1}}{n!}\right)^{t_n}.$$

Moreover,

$$\begin{vmatrix} \frac{F_1}{1!} & 1 & & \\ \frac{F_2}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{F_n}{n!} & \cdots & \frac{F_2}{2!} & \frac{F_1}{1!} \end{vmatrix} = \frac{(-1)^{n-1}}{n!},$$

and

$$\begin{bmatrix} 1 & & & & \\ \frac{F_1}{1!} & 1 & & & \\ \frac{F_2}{2!} & \frac{F_1}{1!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{F_n}{n!} & \cdots & \frac{F_2}{2!} & \frac{F_1}{1!} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & & \\ \frac{1}{1!} & 1 & & & \\ -\frac{1}{2!} & \frac{1}{1!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{(-1)^{n-1}}{n!} & \cdots & -\frac{1}{2!} & \frac{1}{1!} & 1 \end{bmatrix}.$$

Example. For $n = 4$, we have the integer partitions 1+1+1+1, 1+1+2, 1+3, 2+2 and 4. Then

$$
\begin{aligned}
F_4 = 4!\Bigg(& (-1)^{4-4}\binom{4+0+0+0}{4,0,0,0}\left(\frac{1}{1!}\right)^4 \\
& + (-1)^{4-2-1}\binom{2+1+0+0}{2,1,0,0}\left(\frac{1}{1!}\right)^2\left(-\frac{1}{2!}\right) \\
& + (-1)^{4-1-1}\binom{1+0+1+0}{1,0,1,0}\left(\frac{1}{1!}\right)\left(\frac{1}{3!}\right) \\
& + (-1)^{4-2}\binom{0+2+0+0}{0,2,0,0}\left(-\frac{1}{2!}\right)^2 + (-1)^{4-1}\binom{0+0+0+1}{0,0,0,1}\left(-\frac{1}{4!}\right)\Bigg) \\
& = 4!\left(1+\frac{3}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{24}\right) = 75.
\end{aligned}
$$

3 Restricted Fubini numbers

The *restricted Fubini numbers* are defined by

$$F_{n,\leq m} = \sum_{k=0}^{n} k! S_2\left(n,k\right)_{\leq m}\ .$$

They satisfy the recurrence relation:

$$F_{n,\leq m} = \sum_{j=1}^{m}\binom{n}{j} F_{n-j,\leq m}\,. \tag{9}$$

Theorem 6. *The (exponential) generating function of restricted Fubini numbers is given by*

$$\frac{1}{1-t-\frac{t^2}{2!}-\cdots-\frac{t^m}{m!}} = \sum_{n=0}^{\infty} F_{n,\leq m}\frac{t^n}{n!}\,.$$

Proof. Since the generating function of the restricted Stirling numbers of the second kind is given by

$$\frac{1}{k!}\left(t+\frac{t^2}{2}+\cdots+\frac{t^m}{m!}\right)^k = \sum_{n=k}^{mk} S_2\left(n,k\right)_{\leq m}\frac{t^n}{n!}\,,$$

we have

$$\sum_{n=0}^{\infty} F_{n,\leq m}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} k! S_2\left(n,k\right)_{\leq m}\frac{t^n}{n!} = \sum_{k=0}^{\infty} k!\sum_{n=k}^{\infty} S_2\left(n,k\right)_{\leq m}\frac{t^n}{n!}$$

$$= \sum_{k=0}^{\infty} k! \frac{1}{k!} \left(t + \frac{t^2}{2} + \cdots + \frac{t^m}{m!} \right)^k = \frac{1}{1 - t - \frac{t^2}{2} - \cdots - \frac{t^m}{m!}} .$$

□

Theorem 7. *For $n \geq 1$,*

$$F_{n,\leq m} = n! \begin{vmatrix} \frac{1}{1!} & 1 & 0 & & & & 0 \\ -\frac{1}{2!} & \frac{1}{1!} & 1 & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ \frac{(-1)^{m-1}}{m!} & \frac{(-1)^{m-2}}{(m-1)!} & \frac{(-1)^{m-3}}{(m-2)!} & \cdots & & & \\ 0 & \frac{(-1)^{m-1}}{m!} & & & & 1 & 0 \\ \vdots & & & & & \frac{1}{1!} & 1 \\ 0 & \cdots & 0 & \frac{(-1)^{m-1}}{m!} & \cdots & -\frac{1}{2!} & \frac{1}{1!} \end{vmatrix} .$$

Namely, all the values $\frac{(-1)^{j-1}}{j!}$ $(m+1 \leq j \leq n)$ in the determinant in Theorem 2 are replaced by 0.

Remark. When $m \geq n$, the determinant in Theorem 7 matches that in Theorem 2. In fact, $F_{n,\leq m} = F_n$ when $m \geq n$.

Proof of Theorem 7. The result is trivial for $n = 1$, because $F_{1,\leq m} = 1$ $(m \geq 1)$. Assume that the result is true, up to $n - 1$. Then, by expanding the determinant at the first row,

$$\begin{vmatrix} \frac{1}{1!} & 1 & 0 & & & & 0 \\ -\frac{1}{2!} & \frac{1}{1!} & 1 & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ \frac{(-1)^{m-1}}{m!} & \frac{(-1)^{m-2}}{(m-1)!} & \frac{(-1)^{m-3}}{(m-2)!} & \cdots & & & \\ 0 & \frac{(-1)^{m-1}}{m!} & & & & 1 & 0 \\ \vdots & & & & & \frac{1}{1!} & 1 \\ 0 & \cdots & 0 & \frac{(-1)^{m-1}}{m!} & \cdots & -\frac{1}{2!} & \frac{1}{1!} \end{vmatrix}$$

$$= \frac{1}{1!}\frac{F_{n-1,\le m}}{(n-1)!} - \begin{vmatrix} -\frac{1}{2!} & 1 & 0 & & & & 0 \\ \frac{1}{3!} & \frac{1}{1!} & 1 & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ \frac{(-1)^{m-1}}{m!} & \frac{(-1)^{m-2}}{(m-1)!} & \frac{(-1)^{m-3}}{(m-2)!} & \cdots & & & \\ 0 & \frac{(-1)^{m-1}}{m!} & & & & 1 & 0 \\ \vdots & & & & & \frac{1}{1!} & 1 \\ 0 & \cdots & 0 & \frac{(-1)^{m-1}}{m!} & \cdots & -\frac{1}{2!} & \frac{1}{1!} \end{vmatrix}$$

$$= \frac{1}{1!}\frac{F_{n-1,\le m}}{(n-1)!} + \frac{1}{2!}\frac{F_{n-2,\le m}}{(n-2)!} + \cdots$$

$$+ \frac{1}{m!}\frac{F_{n-m,\le m}}{(n-m)!} - \begin{vmatrix} 0 & 1 & 0 & & & & 0 \\ 0 & \frac{1}{1!} & 1 & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ 0 & \frac{(-1)^{m-2}}{(m-1)!} & \frac{(-1)^{m-3}}{(m-2)!} & \cdots & & & \\ 0 & \frac{(-1)^{m-1}}{m!} & & & & 1 & 0 \\ \vdots & & & & & \frac{1}{1!} & 1 \\ 0 & \cdots & 0 & \frac{(-1)^{m-1}}{m!} & \cdots & -\frac{1}{2!} & \frac{1}{1!} \end{vmatrix}$$

$$= \sum_{j=1}^{m} \frac{1}{j!}\frac{F_{n-j,\le m}}{(n-j)!} = \frac{F_{n,\le m}}{n!}\,.$$

Here, we used the relation (9). □

Example. By Theorem 2,

$$6!\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 1 & 0 & 0 \\ \frac{1}{6} & -\frac{1}{2} & 1 & 1 & 0 \\ -\frac{1}{24} & \frac{1}{6} & -\frac{1}{2} & 1 & 1 \\ \frac{1}{120} & -\frac{1}{24} & \frac{1}{6} & -\frac{1}{2} & 1 \end{vmatrix} = 541$$

and

$$5!\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \\ -\frac{1}{2} & 1 & 1 & 0 & 0 & 0 \\ \frac{1}{6} & -\frac{1}{2} & 1 & 1 & 0 & 0 \\ -\frac{1}{24} & \frac{1}{6} & -\frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{120} & -\frac{1}{24} & \frac{1}{6} & -\frac{1}{2} & 1 & 1 \\ -\frac{1}{720} & \frac{1}{120} & -\frac{1}{24} & \frac{1}{6} & -\frac{1}{2} & 1 \end{vmatrix} = 4683\,.$$

Since

$$\frac{1}{2-e^t} = 1+t+\frac{3t^2}{2}+\frac{13t^3}{6}+\frac{25t^4}{8}+\frac{541t^5}{120}+\frac{1561t^6}{240}+\frac{47293t^7}{5040}+\frac{36389t^8}{2688}+\cdots$$

we also have

$$F_5 = \frac{541}{120}5! = 541 \quad \text{and} \quad F_6 = \frac{1561}{240}6! = 4683\,.$$

On the other hand, by Theorem 7,

$$5!\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 1 & 0 & 0 \\ \frac{1}{6} & -\frac{1}{2} & 1 & 1 & 0 \\ 0 & \frac{1}{6} & -\frac{1}{2} & 1 & 1 \\ 0 & 0 & \frac{1}{6} & -\frac{1}{2} & 1 \end{vmatrix} = 530$$

and

$$6!\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \\ -\frac{1}{2} & 1 & 1 & 0 & 0 & 0 \\ \frac{1}{6} & -\frac{1}{2} & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{6} & -\frac{1}{2} & 1 & 1 & 0 \\ 0 & 0 & \frac{1}{6} & -\frac{1}{2} & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{6} & -\frac{1}{2} & 1 \end{vmatrix} = 4550\,.$$

Since

$$\frac{1}{1-(t+\frac{t^2}{2}+\frac{t^3}{6})} = 1+t+\frac{3t^2}{2}+\frac{13t^3}{6}+\frac{37t^4}{12}+\frac{53t^5}{12}+\frac{455t^6}{72}+\frac{217t^7}{24}+\frac{207t^8}{16}+\cdots$$

we also have

$$F_{5,\le 3} = \frac{53}{12}5! = 530 \quad \text{and} \quad F_{6,\le 3} = \frac{455}{72}6! = 4550\,.$$

From Trudi's formula and Lemma 4 we obtain the following relations.

Corollary 8. *For $n \ge 1$*

$$\frac{F_{n,\le m}}{n!} = \sum_{t_1+2t_2+\cdots+mt_m=n} \binom{t_1+\cdots+t_m}{t_1,\ldots,t_m} \times (-1)^{n-t_1-\cdots-t_m} \left(\frac{1}{1!}\right)^{t_1} \left(\frac{(-1)^1}{2!}\right)^{t_2} \cdots \left(\frac{(-1)^{m-1}}{m!}\right)^{t_m}.$$

Moreover,

$$\begin{vmatrix} \frac{F_{1,\leq m}}{1!} & 1 & & \\ \frac{F_{2,\leq m}}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{F_{n,\leq m}}{n!} & \ddots & \frac{F_{2,\leq m}}{2!} & \frac{F_{1,\leq m}}{1!} \end{vmatrix} = 0,$$

and

$$\begin{bmatrix} 1 & & & & \\ \frac{F_{1,\leq m}}{1!} & 1 & & & \\ \frac{F_{2,\leq m}}{2!} & \frac{F_{1,\leq m}}{1!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{F_{n,\leq m}}{n!} & \cdots & \frac{F_{2,\leq m}}{2!} & \frac{F_{1,\leq m}}{1!} & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & & & & & & & & \\ \frac{1}{1!} & 1 & & & & & & & \\ -\frac{1}{2!} & \frac{1}{1!} & 1 & & & & & & \\ \vdots & & \ddots & & & & & & \\ \frac{(-1)^{m-1}}{m!} & \frac{(-1)^{m-2}}{(m-1)!} & \frac{(-1)^{m-3}}{(m-2)!} & \cdots & & & & & \\ 0 & \frac{(-1)^{m-1}}{m!} & & \cdots & & 1 & & & \\ \vdots & & & \cdots & & \frac{1}{1!} & 1 & & \\ 0 & & & \cdots & \frac{(-1)^{m-1}}{m!} & \cdots & \frac{1}{1!} & 1 & \end{bmatrix}.$$

4 Associated Fubini numbers

The *associated Fubini numbers* are defined by

$$F_{n,\geq m} = \sum_{k=0}^{n} k! S_2\left(n,k\right)_{\geq m} .$$

They satisfy the recurrence relation:

$$F_{n,\geq m} = \sum_{j=m}^{n} \binom{n}{j} F_{n-j,\geq m} . \tag{10}$$

Theorem 9. *The (exponential) generating function of restricted Fubini numbers is given by*

$$\frac{1}{1-\frac{t^m}{m!}-\frac{t^{m+1}}{(m+1)!}-\cdots} = \sum_{n=0}^{\infty} F_{n,\geq m} \frac{t^n}{n!} .$$

Proof. Since the generating function of the associated Stirling numbers of the second kind is given by

$$\frac{1}{k!}\left(\frac{t^m}{m!}+\frac{t^{m+1}}{(m+1)!}+\cdots\right)^k=\sum_{n=k}^{mk} S_2\left(n,k\right)_{\geq m}\frac{t^n}{n!},$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,\geq m}\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\sum_{k=0}^{n} k!S_2\left(n,k\right)_{\geq m}\frac{t^n}{n!}=\sum_{k=0}^{\infty} k!\sum_{n=k}^{\infty} S_2\left(n,k\right)_{\geq m}\frac{t^n}{n!}\\&=\sum_{k=0}^{\infty} k!\frac{1}{k!}\left(\frac{t^m}{m!}+\frac{t^{m+1}}{(m+1)!}+\cdots\right)^k=\frac{1}{1-\frac{t^m}{m!}-\frac{t^{m+1}}{(m+1)!}-\cdots}.\end{aligned}$$

□

Theorem 10. *For $n\geq m\geq 1$,*

$$F_{n,\geq m}=n!\begin{vmatrix} 0 & 1 & 0 & \cdots & & & 0\\ \vdots & 0 & 1 & 0 & \cdots & & \vdots\\ 0 & \vdots & \ddots & \ddots & & & \\ \frac{(-1)^{m-1}}{m!} & 0 & & & & & \vdots\\ \vdots & \ddots & & & & \ddots & 0\\ \frac{(-1)^{n-2}}{(n-1)!} & & & \ddots & & 0 & 1\\ \frac{(-1)^{n-1}}{n!} & \frac{(-1)^{n-2}}{(n-1)!} & \cdots & \frac{(-1)^{m-1}}{m!} & 0 & \cdots & 0\end{vmatrix}.$$

Namely, all the values $\frac{(-1)^{j-1}}{j!}$ $(1\leq j\leq m-1)$ in the determinant in Theorem 2 are replaced by 0.

Remark. When $m=1$, the determinant in Theorem 10 matches that in Theorem 2. In fact, $F_{n,\geq 1}=F_n$.

Proof of Theorem 10. For convenience, put

$$G_{n,m} := \begin{vmatrix} 0 & 1 & 0 & \cdots & & & 0 \\ \vdots & 0 & 1 & 0 & \cdots & & \vdots \\ 0 & \vdots & \ddots & \ddots & & & \\ \frac{(-1)^{m-1}}{m!} & 0 & & & & & \vdots \\ \vdots & \ddots & & & & \ddots & 0 \\ \frac{(-1)^{n-2}}{(n-1)!} & & & \ddots & & 0 & 1 \\ \frac{(-1)^{n-1}}{n!} & \frac{(-1)^{n-2}}{(n-1)!} & \cdots & \frac{(-1)^{m-1}}{m!} & 0 & \cdots & 0 \end{vmatrix}.$$

Expanding the determinant at the first row of $G_{n,m}$ continually, if $n < 2m$, then

$$\begin{aligned} G_{n,m} &= (-1)^{m-1} \begin{vmatrix} \frac{(-1)^{m-1}}{(m-1)!} & 1 & & \\ \vdots & & & \\ \vdots & & & 1 \\ \frac{(-1)^{n-1}}{n!} & 0 & \cdots & 0 \end{vmatrix} \\ &= (-1)^{m-1}(-1)^{n-m}\frac{(-1)^{n-1}}{n!} = \frac{1}{n!}. \end{aligned}$$

Since $F_{n,\geq m} = 1$ $(n < 2m)$, we have $F_{n,\geq m} = n!G_{n,m}$, as desired.

If $n \geq 2m$, then

$$\begin{aligned} G_{n,m} &= (-1)^{m-1} \begin{vmatrix} \frac{(-1)^{m-1}}{m!} & 1 & & \\ \vdots & & & \\ \vdots & & & 1 \\ \frac{(-1)^{n-1}}{n!} & \frac{(-1)^{n-m+1}}{(n-m)!} & & 0 \end{vmatrix} \\ &= \frac{1}{m!} \begin{vmatrix} 0 & 1 & 0 & & & \\ \vdots & & & & & \\ 0 & & & & & \\ \frac{(-1)^{m-1}}{m!} & & & & & 0 \\ \vdots & & & & & 1 \\ \frac{(-1)^{n-m+1}}{(n-m)!} & \cdots & \frac{(-1)^{m-1}}{m!} & 0 & \cdots & 0 \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
&-(-1)^{m-1}\begin{vmatrix} \frac{(-1)^m}{(m+1)!} & 1 & & \\ \vdots & & & \\ \vdots & & & 1 \\ \frac{(-1)^{n-1}}{n!} & \frac{(-1)^{n-m}}{(n-m-1)!} & & 0 \end{vmatrix} \\
&= \frac{1}{m!}\frac{F_{n-m,\geq m}}{(n-m)!} + \frac{1}{(m+1)!}\begin{vmatrix} 0 & 1 & 0 & & & \\ \vdots & & & & & \\ 0 & & & & & \\ \frac{(-1)^{m-1}}{m!} & & & & & 0 \\ \vdots & & & & & 1 \\ \frac{(-1)^{n-m}}{(n-m-1)!} & \cdots & \frac{(-1)^{m-1}}{m!} & 0 & \cdots & 0 \end{vmatrix} \\
&-(-1)^{m}\begin{vmatrix} \frac{(-1)^{m+1}}{(m+2)!} & 1 & & \\ \vdots & & & \\ \vdots & & & 1 \\ \frac{(-1)^{n-1}}{n!} & \frac{(-1)^{n-m+1}}{(n-m-2)!} & & 0 \end{vmatrix} = \sum_{j=m}^{n}\frac{1}{j!}\frac{F_{n-j,\geq m}}{(n-j)!} = \frac{F_{n,\geq m}}{n!}\,.
\end{aligned}
$$

Here, we use the relation (10). □

Example.

By Theorem 10,

$$
5!\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 1 & 0 \\ -\frac{1}{24} & \frac{1}{6} & 0 & 0 & 1 \\ \frac{1}{120} & -\frac{1}{24} & \frac{1}{6} & 0 & 0 \end{vmatrix} = 1
$$

and

$$
6!\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{24} & \frac{1}{6} & 0 & 0 & 1 & 0 \\ \frac{1}{120} & -\frac{1}{24} & \frac{1}{6} & 0 & 0 & 1 \\ -\frac{1}{720} & \frac{1}{120} & -\frac{1}{24} & \frac{1}{6} & 0 & 0 \end{vmatrix} = 21\,.
$$

Since

$$
\frac{1}{1-(\frac{t^3}{6}+\frac{t^4}{24}+\cdots)} = 1+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+\frac{7t^6}{240}+\frac{61t^8}{13440}+\frac{2101t^9}{362880}+\cdots,
$$

we also have

$$F_{5,\geq 3} = \frac{1}{6}6! = 1 \quad \text{and} \quad F_{6,\geq 3} = \frac{7}{240}6! = 21\,.$$

From Trudi's formula and Lemma 4 we obtain the following relations.

Corollary 11. *For $n \geq 1$*

$$\frac{F_{n,\geq m}}{n!} = \sum_{mt_m+(m+1)t_{m+1}+\cdots+nt_n=n} \binom{t_m+\cdots+t_n}{t_m,\ldots,t_n} (-1)^{n-t_m-\cdots-t_n} \times \left(\frac{(-1)^{m-1}}{m!}\right)^{t_m} \left(\frac{(-1)^m}{(m+1)!}\right)^{t_{m+1}} \cdots \left(\frac{(-1)^{n-1}}{n!}\right)^{t_n}. \quad (11)$$

Moreover,

$$\begin{vmatrix} \frac{F_{1,\geq m}}{1!} & 1 & & \\ \frac{F_{2,\geq m}}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{F_{n,\geq m}}{n!} & \ddots & \frac{F_{2,\geq m}}{2!} & \frac{F_{1,\geq m}}{1!} \end{vmatrix} = \frac{(-1)^{n-1}}{n!},$$

and

$$\begin{bmatrix} 1 & & & \\ \frac{F_{1,\geq m}}{1!} & 1 & & \\ \frac{F_{2,\geq m}}{2!} & \frac{F_{1,\geq m}}{1!} & 1 & \\ \vdots & & \ddots & \\ \frac{F_{n,\geq m}}{n!} & \cdots & \frac{F_{2,\geq m}}{2!} & \frac{F_{1,\geq m}}{1!} & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ \vdots & & \ddots & & & & & \\ \frac{(-1)^{m-1}}{m!} & 0 & & \cdots & & & \vdots & \\ \vdots & \ddots & & & & \ddots & 0 & \\ \frac{(-1)^{n-2}}{(n-1)!} & & & \cdots & & 0 & 1 & \\ \frac{(-1)^{n-1}}{n!} & \frac{(-1)^{n-2}}{(n-1)!} & \cdots & \frac{(-1)^{m-1}}{m!} & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

5 Modified incomplete Bernoulli and Cauchy numbers

In [9, 11], restricted Cauchy numbers $c_{n,\le m}$ and associated Cauchy numbers $c_{n,\ge m}$ are introduced as

$$\sum_{n=0}^{\infty} c_{n,\le m}\frac{t^n}{n!} = \frac{e^{F_m(t)}-1}{F_m(t)} \tag{12}$$

and

$$\sum_{n=0}^{\infty} c_{n,\ge m}\frac{t^n}{n!} = \frac{e^{\ln(1+t)-F_{m-1}(t)}-1}{\ln(1+t)-F_{m-1}(t)}\,, \tag{13}$$

respectively, where

$$F_m(t) = t - \frac{t^2}{2} + \cdots + (-1)^{m-1}\frac{t^m}{m} \quad (m \ge 1)\,.$$

Both incomplete Cauchy numbers are natural extensions of the original Cauchy numbers c_n because they have expressions:

$$c_{n,\le m} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\le m} \frac{(-1)^{n-k}}{k+1}$$

and

$$c_{n,\ge m} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\ge m} \frac{(-1)^{n-k}}{k+1}$$

with $c_n = c_{n,\le\infty} = c_{n,\ge 1}$. However, such incomplete Cauchy numbers cannot be natural extensions of the original one in terms of the determinants, though Fubini numbers do as seen in Theorems 2, 7 and 10. Therefore, we introduced *modified* incomplete Cauchy numbers. The *modified restricted Cauchy numbers* $c^*_{n,\le m}$ are defined by

$$\frac{t}{F_m(t)} = \sum_{n=0}^{\infty} c^*_{n,\le m}\frac{t^n}{n!} \quad (m \ge 2) \tag{14}$$

instead of (12). The *modified associated Cauchy numbers* $c^*_{n,\ge m}$ are defined by

$$\frac{t}{\ln(1+t)-F_{m-1}(t)+t} = \sum_{n=0}^{\infty} c^*_{n,\ge m}\frac{t^n}{n!} \quad (m \ge 2) \tag{15}$$

instead of (13). Note that $c_n = c^*_{n,\le m}$ if $n \le m-1$, and $c_n = c^*_{n,\ge 2}$.

Theorem 12. *For integers $n \geq 1$ and $m \geq 2$,*

$$c^*_{n,\leq m} = n! \begin{vmatrix} \frac{1}{2} & 1 & 0 & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ \frac{1}{m} & & & & & \\ 0 & \ddots & & & & 0 \\ & & & & & 1 \\ & & 0 & \frac{1}{m} & \cdots & \frac{1}{2} \end{vmatrix}.$$

Theorem 13. *For integers n and m with $n-1 \geq m \geq 2$,*

$$c^*_{n,\geq m} = n! \begin{vmatrix} 0 & 1 & 0 & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ \frac{1}{m} & & & & \ddots & \\ \frac{1}{m+1} & \ddots & & & \ddots & 0 \\ \vdots & & \ddots & & \ddots & 1 \\ \frac{1}{n+1} & \cdots & \frac{1}{m+1} & \frac{1}{m} & \cdots & 0 \end{vmatrix}.$$

Remark. If $n \leq m-1$, this result is reduced to (4).

Proof of Theorem 12. By (14),

$$1 = \left(\sum_{j=0}^{m-1} (-1)^j \frac{t^j}{j+1} \right) \left(\sum_{l=0}^{\infty} c^*_{l,\leq m} \frac{t^l}{l!} \right) = \sum_{n=0}^{\infty} \sum_{j=0}^{\min\{n,m-1\}} \frac{(-1)^j}{j+1} \frac{c^*_{n-j,\leq m}}{(n-j)!} t^n .$$

Hence,

$$\frac{c^*_{n,\leq m}}{n!} = \sum_{j=1}^{m} \frac{(-1)^{j+1}}{j+1} \frac{c^*_{n-j,\leq m}}{(n-j)!} \quad (n \geq 1). \tag{16}$$

By (14), it is clear that $c^*_{1,\leq m} = \frac{1}{2}$. Assume that the result is valid up to $n-1$. Then, by expanding the determinant at the first row, we have

$$\begin{vmatrix} \frac{1}{2} & 1 & 0 & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ \frac{1}{m} & & & & & \\ 0 & \ddots & & & & 0 \\ & & & & & 1 \\ & & 0 & \frac{1}{m} & \cdots & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \frac{c^*_{n-1,\leq m}}{(n-1)!} - \begin{vmatrix} \frac{1}{3} & 1 & 0 & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ \frac{1}{m} & & & & & \\ 0 & \ddots & & & & 0 \\ & & & & & 1 \\ & & 0 & \frac{1}{m} & \cdots & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}\frac{c^*_{n-1,\le m}}{(n-1)!} - \frac{1}{3}\frac{c^*_{n-2,\le m}}{(n-2)!} + \cdots + (-1)^m \begin{vmatrix} \frac{1}{m} & 1 & 0 & \\ 0 & \frac{1}{2} & & \\ \vdots & & & 0 \\ \vdots & & & 1 \\ 0 & & & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}\frac{c^*_{n-1,\le m}}{(n-1)!} - \frac{1}{3}\frac{c^*_{n-2,\le m}}{(n-2)!} + \cdots + \frac{(-1)^m}{m}\frac{c^*_{n-m+1,\le m}}{(n-m+1)!} = \frac{c^*_{n,\le m}}{n!}\,.$$

Here, we used the relation (16). □

Remark. If $m = 2$, this result is reduced to (4).

Proof of Theorem 13. If $n+1 < 2m$, the identity is equivalent to

$$\frac{c^*_{n,\ge m}}{n!} = (-1)^m \begin{vmatrix} \frac{1}{m} & 1 & 0 & & & & \\ \vdots & 0 & \ddots & & & & \\ & \vdots & & & & & \\ & 0 & & & & & \\ & \frac{1}{m} & & & & \ddots & 0 \\ \vdots & \vdots & \ddots & & & 0 & 1 \\ \frac{1}{n+1} & \frac{1}{n-m+2} & \cdots & \frac{1}{m} & 0 & \cdots & 0 \end{vmatrix}.$$

If $n+1 \ge 2m$, the identity is equivalent to

$$\frac{c^*_{n,\ge m}}{n!} = (-1)^m \begin{vmatrix} \frac{1}{m} & 1 & 0 & \\ \vdots & 0 & & 0 \\ \vdots & \vdots & \ddots & 1 \\ \frac{1}{n+1} & 0 & \cdots & 0 \end{vmatrix}.$$

By (15), we have

$$1 = \left(1 + \sum_{j=m-1}^{\infty} (-1)^j \frac{t^j}{j+1}\right)\left(\sum_{l=0}^{\infty} c^*_{l,\ge m}\frac{t^l}{l!}\right)$$
$$= \sum_{n=0}^{\infty} c^*_{n,\ge m}\frac{t^n}{n!} + \sum_{n=0}^{\infty}\sum_{j=m-1}^{n} \frac{(-1)^j}{j+1}\frac{c^*_{n-j,\ge m}}{(n-j)!}t^n\,.$$

Hence, for $n \geq 1$

$$\frac{c^*_{n,\geq m}}{n!} = \sum_{j=m-1}^{n} \frac{(-1)^{j+1}}{j+1} \frac{c^*_{n-j,\geq m}}{(n-j)!}\,.$$

By using this relation, we can obtain the result, similarly to the proof of Theorem 12. □

For the modified Cauchy numbers, we can also give an explicit expression similar to that given in Corollaries 5, 8 and 11.

Corollary 14. *For $n \geq 1$*

$$\frac{c^*_{n,\leq m}}{n!} = \sum_{t_1+2t_2+\cdots+(m-1)t_{m-1}=n} \binom{t_1+\cdots+t_{m-1}}{t_1,\ldots,t_{m-1}} \times (-1)^{n-t_1-\cdots-t_{m-1}} \left(\frac{1}{2}\right)^{t_1} \left(\frac{1}{3}\right)^{t_2} \cdots \left(\frac{1}{m}\right)^{t_{m-1}}.$$

Moreover,

$$\begin{vmatrix} \frac{c^*_{1,\leq m}}{1!} & 1 & & \\ \frac{c^*_{2,\leq m}}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{c^*_{n,\leq m}}{n!} & \ddots & \frac{c^*_{2,\leq m}}{2!} & \frac{c^*_{1,\leq m}}{1!} \end{vmatrix} = 0,$$

and

$$\begin{bmatrix} 1 & & & & \\ \frac{c^*_{1,\leq m}}{1!} & 1 & & & \\ \frac{c^*_{2,\leq m}}{2!} & \frac{c^*_{1,\leq m}}{1!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{c^*_{n,\leq m}}{n!} & \cdots & \frac{c^*_{2,\leq m}}{2!} & \frac{c^*_{1,\leq m}}{1!} & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & & & & & & & \\ \frac{1}{2} & 1 & & & & & & \\ \frac{1}{3} & \frac{1}{2} & 1 & & & & & \\ \vdots & & \ddots & & & & & \\ \frac{1}{m} & \frac{1}{m-1} & \frac{1}{m-2} & \cdots & & & & \\ 0 & \frac{1}{m} & & \cdots & & 1 & & \\ \vdots & & & \cdots & & \frac{1}{2} & 1 & \\ 0 & & & \cdots & \frac{1}{m} & \cdots & \frac{1}{2} & 1 \end{bmatrix}.$$

Corollary 15. *For $n \geq 1$*

$$\frac{c^*_{n,\geq m}}{n!} = \sum_{mt_{m-1}+(m+1)t_m+\cdots+(n+1)t_n=n} \binom{t_{m-1}+\cdots+t_n}{t_{m-1},\ldots,t_n} \times (-1)^{n-t_{m-1}-\cdots-t_n} \left(\frac{1}{m}\right)^{t_{m-1}} \left(\frac{1}{m+1}\right)^{t_m} \cdots \left(\frac{1}{n+1}\right)^{t_n}. \quad (17)$$

Moreover,

$$\begin{vmatrix} \frac{c^*_{1,\geq m}}{1!} & 1 & & \\ \frac{c^*_{2,\geq m}}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{c^*_{n,\geq m}}{n!} & \ddots & \frac{c^*_{2,\geq m}}{2!} & \frac{c^*_{1,\geq m}}{1!} \end{vmatrix} = \frac{1}{n+1},$$

and

$$\begin{bmatrix} 1 & & & & \\ \frac{c^*_{1,\geq m}}{1!} & 1 & & & \\ \frac{c^*_{2,\geq m}}{2!} & \frac{c^*_{1,\geq m}}{1!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{c^*_{n,\geq m}}{n!} & \cdots & \frac{c^*_{2,\geq m}}{2!} & \frac{c^*_{1,\geq m}}{1!} & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ \vdots & & & \ddots & & & & \\ \frac{1}{m} & 0 & & \cdots & & & \vdots & \\ \vdots & \ddots & & & & \ddots & & \\ \frac{1}{n} & & & \cdots & & 0 & 1 & \\ \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{m} & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

In [10], restricted Bernoulli numbers $B_{n,\leq m}$ and associated Bernoulli numbers $B_{n,\geq m}$ are introduced. However, similarly to incomplete Cauchy numbers, these incomplete Bernoulli numbers must be modified to have determinant expressions.

Define *modified restricted Bernoulli numbers* $B^*_{n,\leq m}$ by

$$\frac{t}{E_m(t)-1} = \sum_{n=0}^{\infty} B^*_{n,\leq m} \frac{t^n}{n!} \quad (18)$$

and define *modified associated Bernoulli numbers* $B^*_{n,\geq m}$ by

$$\frac{t}{e^t - E_m(t) - 1 + t} = \sum_{n=0}^{\infty} B^*_{n,\geq m} \frac{t^n}{n!}, \tag{19}$$

where

$$E_m(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^m}{m!}.$$

Then these modified incomplete Bernoulli numbers have determinant expressions. The proofs are similar and omitted.

Theorem 16. *For integers $n \geq 1$ and $m \geq 2$,*

$$B^*_{n,\leq m} = (-1)^n n! \begin{vmatrix} \frac{1}{2!} & 1 & 0 & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ \frac{1}{m!} & & & & & \\ 0 & \ddots & & & & 0 \\ & & & & & 1 \\ & & 0 & \frac{1}{m!} & \cdots & \frac{1}{2!} \end{vmatrix}.$$

Remark. If $n \leq m-1$, this result is reduced to (3).

Theorem 17. *For integers n and m with $n - 1 \geq m \geq 2$,*

$$B^*_{n,\geq m} = (-1)^n n! \begin{vmatrix} 0 & 1 & 0 & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ \frac{1}{m!} & & & & \ddots & \\ \frac{1}{(m+1)!} & \ddots & & & \ddots & 0 \\ \vdots & & \ddots & & \ddots & 1 \\ \frac{1}{(n+1)!} & \cdots & \frac{1}{(m+1)!} & \frac{1}{m!} & \cdots & 0 \end{vmatrix}.$$

Remark. If $m = 2$, this result is reduced to (3).

Corollary 18. *For $n \geq 1$*

$$\frac{(-1)^n B^*_{n,\leq m}}{n!} = \sum_{t_1 + 2t_2 + \cdots + (m-1)t_{m-1} = n} \binom{t_1 + \cdots + t_{m-1}}{t_1, \ldots, t_{m-1}}$$

$$\times (-1)^{n-t_1-\cdots-t_{m-1}} \left(\frac{1}{2!}\right)^{t_1} \left(\frac{1}{3!}\right)^{t_2} \cdots \left(\frac{1}{m!}\right)^{t_{m-1}}.$$

Moreover,

$$\begin{vmatrix} \frac{-B^*_{1,\le m}}{1!} & 1 & & \\ \frac{(-1)^2 B^*_{2,\le m}}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{(-1)^n B^*_{n,\le m}}{n!} & \ddots & \frac{(-1)^2 B^*_{2,\le m}}{2!} & \frac{-B^*_{1,\le m}}{1!} \end{vmatrix} = 0,$$

and

$$\begin{bmatrix} 1 & & & & \\ \frac{-B^*_{1,\le m}}{1!} & 1 & & & \\ \frac{(-1)^2 B^*_{2,\le m}}{2!} & \frac{-B^*_{1,\le m}}{1!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{(-1)^n B^*_{n,\le m}}{n!} & \cdots & \frac{(-1)^2 B^*_{2,\le m}}{2!} & \frac{-B^*_{1,\le m}}{1!} & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & & & & & & & \\ \frac{1}{2!} & 1 & & & & & & \\ \frac{1}{3!} & \frac{1}{2!} & 1 & & & & & \\ \vdots & & \ddots & & & & & \\ \frac{1}{m!} & \frac{1}{(m-1)!} & \frac{1}{(m-2)!} & \cdots & & & & \\ 0 & \frac{1}{m!} & & \cdots & 1 & & & \\ \vdots & & & \cdots & & \frac{1}{2!} & 1 & \\ 0 & & & \cdots & \frac{1}{m!} & \cdots & \frac{1}{2!} & 1 \end{bmatrix}.$$

Corollary 19. *For $n \ge 1$*

$$\frac{(-1)^n B^*_{n,\ge m}}{n!} = \sum_{mt_{m-1}+(m+1)t_m+\cdots+(n+1)t_n=n} \binom{t_{m-1}+\cdots+t_n}{t_{m-1},\ldots,t_n}$$
$$\times (-1)^{n-t_{m-1}-\cdots-t_n} \left(\frac{1}{m!}\right)^{t_{m-1}} \left(\frac{1}{(m+1)!}\right)^{t_m} \cdots \left(\frac{1}{(n+1)!}\right)^{t_n}. \quad (20)$$

Moreover,

$$\begin{vmatrix} \frac{-B^*_{1,\geq m}}{1!} & 1 & & \\ \frac{(-1)^2 B^*_{2,\geq m}}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{(-1)^n B^*_{n,\geq m}}{n!} & \ddots & \frac{(-1)^2 B^*_{2,\geq m}}{2!} & \frac{-B^*_{1,\geq m}}{1!} \end{vmatrix} = \frac{1}{(n+1)!},$$

and

$$\begin{bmatrix} 1 & & & & \\ \frac{-B^*_{1,\geq m}}{1!} & 1 & & & \\ \frac{(-1)^2 B^*_{2,\geq m}}{2!} & \frac{-B^*_{1,\geq m}}{1!} & 1 & & \\ \vdots & & \ddots & & \\ \frac{(-1)^n B^*_{n,\geq m}}{n!} & \cdots & \frac{(-1)^2 B^*_{2,\geq m}}{2!} & \frac{-B^*_{1,\geq m}}{1!} & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ \vdots & & & \ddots & & & & \\ \frac{1}{m!} & 0 & & \cdots & & & \vdots & \\ \vdots & \ddots & & & & \ddots & & \\ \frac{1}{n!} & & & \cdots & & 0 & 1 & \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{m!} & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

6 The Generalized Fubini Numbers

The *generalized Fubini numbers* $F_n^{(k)}$ are defined by

$$\left(\frac{1}{2-e^t}\right)^k = \sum_{j=0}^{\infty} F_{j+1}^{(k)} \frac{t^j}{j!},\ k \in \mathbb{Z}^+.$$

Note that

$$\frac{F_{n+1}^{(k)}}{n!} = \sum_{j_1+j_2+\cdots+j_k=n} \frac{F_{j_1+1}}{(j_1+1)!} \frac{F_{j_2+1}}{(j_2+1)!} \cdots \frac{F_{j_k+1}}{(j_k+1)!}. \tag{21}$$

The generating functions of the generalized Fubini numbers for $k = 2$ and 3 are

$$\left(\frac{1}{2-e^t}\right)^2 = 1 + 2t + 4t^2 + \frac{22t^3}{3} + \frac{77t^4}{6} + \frac{653t^5}{30} + \frac{6497t^6}{180} + \frac{74141t^7}{1260} + \cdots$$
$$\left(\frac{1}{2-e^t}\right)^3 = 1 + 3t + \frac{15t^2}{2} + \frac{33t^3}{2} + \frac{269t^4}{8} + \frac{2601t^5}{40} + \frac{5809t^6}{48} + \cdots$$

Let $T_n = [t_{i,j}]$ be the n-square Toeplitz-Hessenberg matrix defined as in Theorem 2, i.e.,

$$t_{i,j} = t_{i-j} = \begin{cases} \frac{(-1)^{i-1}}{i!}, & \text{if } i \geq j; \\ 1, & \text{if } i+1 = j; \\ 0, & \text{otherwise.} \end{cases}$$

Let A and C be matrices of size $n \times n$ and $m \times m$, respectively, and B be a $n \times m$ matrix. Since

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det A \det C,$$

the principal minor $M(i)$ of the matrix T_n is equal to $\frac{F_i}{i!}\frac{F_{n-i+1}}{(n-i+1)!}$. It follows that the principal minor $M(i_1, i_2, \ldots, i_l)$ of the matrix T_n is obtained by deleting rows and columns with indices $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n$:

$$M(i_1, i_2, \ldots, i_l) = \frac{F_{i_1}}{i_1!}\frac{F_{i_2-i_1}}{(i_2-i_1)!}\cdots\frac{F_{i_l-i_{l-1}}}{(i_l-i_{l-1})!}\frac{F_{n-i_l+1}}{(n-i_l+1)!}. \tag{22}$$

Then from (22) we have the following theorem.

Theorem 20. *Let $S_{n-\ell}, (\ell = 0, 1, 2, \ldots, n-1)$ be the sum of all principal minors of the matrix T_n of order $n - \ell$. Then*

$$S_{n-\ell} = \sum_{j_1+j_2+\cdots+j_{\ell+1}=n-\ell} \frac{F_{j_1}+1}{(j_1+1)!}\frac{F_{j_2}+1}{(j_2+1)!}\cdots\frac{F_{j_{\ell+1}}+1}{(j_{\ell+1}+1)!}. \tag{23}$$

Since the coefficients of the characteristic polynomial of a matrix are, up to the sign, sums of principal minors of the matrix, then we have the following.

Corollary 21. *The generalized Fubini number $\frac{F_{n-\ell+1}^{(\ell+1)}}{(n-\ell+1)!}$ is equal, up to the sign, to the coefficient of x^ℓ in the characteristic polynomial $p_n(x)$ of the matrix T_n*

For example,

$$T_5 = \begin{bmatrix} \frac{1}{1!} & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{1!} & 1 & 0 & 0 \\ \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} & 1 & 0 \\ -\frac{1}{4!} & \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} & 1 \\ \frac{1}{5!} & -\frac{1}{4!} & \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} \end{bmatrix}$$

then its characteristic polynomial is $p_5(x) = -x^5+5x^4-12x^3+\frac{33x^2}{2}-\frac{77x}{6}+\frac{541}{120}$. So, it is clear that the coefficient of x^1 is $\frac{F_5^{(2)}}{5!} = \frac{77}{6}$. Note that this is the 4-th coefficient of the ordinary generating function of $(\frac{1}{2-e^t})^2$.

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